

# A gap in the paper "A note on cone metric fixed point theory and its equivalence" [Nonlinear Anal. 72(5), (2010), 2259-2261]

Thabet ABDELJAWAD\*, Erdal KARAPINAR†

## Abstract

There is a gap in Theorem 2.2 of the paper of Du ([1]). In this paper, we shall state the gap and repair it.

In 2010, Du investigated the equivalence of vectorial versions of fixed point theorems in generalized cone metric spaces and scalar versions of fixed point theorems in (general) metric spaces (in usual sense). He showed that the Banach contraction principles in general metric spaces and in TVS-cone metric spaces are equivalent. His results also extended some results of [2] and [4]. In this paper, all notations are considered as in [1]. Further,  $(E; S)$  will stand for the Hausdorff locally convex topological vector space with  $S$  the system of seminorms generating its topology. Also we insist on that continuity of the algebraic operations in a topological vector space and the properties of the cone imply the relations:

$$\text{int}P + \text{int}P \subset \text{int}P \quad \text{and} \quad \lambda \text{int}P \subset \text{int}P \quad \text{for each} \quad \lambda > 0.$$

We appeal to these relations in the following. Du proved the following result [[1]; Theorem 2.2].

**Theorem 1.** *Let  $(X, p)$  be a TVS-CMS,  $x \in X$  and  $\{x_n\}_{n=1}^{\infty}$  a sequence in  $X$ . Set  $d_p = \xi_e \circ p$ . Then the following statements hold:*

- (i) *If  $\{x_n\}_{n=1}^{\infty}$  converges to  $x$  in TVS-CMS  $(X, p)$ , then  $d_p(x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$ ,*
- (ii) *If  $\{x_n\}_{n=1}^{\infty}$  is a Cauchy sequence in TVS-CMS  $(X, p)$ , then  $\{x_n\}_{n=1}^{\infty}$  is a Cauchy sequence (in usual sense) in  $(X, d_p)$ ,*
- (iii) *If  $(X, p)$  is a complete TVS-CMS, then  $(X, d_p)$  is a complete metric space.*

---

\*Çankaya University, Department of Mathematics, 06530, Ankara, Turkey

†Atılım University, Department of Mathematics, İncek 06836, Ankara, Turkey

The author has been claimed that the conclusion (iii) is immediate from conditions (i) and (ii). This assertion is not true. Take a Cauchy sequence  $\{x_n\}$  in  $(X, d_p)$ . To proceed by (ii), one needs to show  $\{x_n\}$  is Cauchy in  $(X, p)$ , which means that the converse of the statement (ii) must also hold.

In fact, the converse of the implications of (i) and (ii) hold. We prove it here. Regarding (i) we prove that if  $x_n \rightarrow x$  in  $(X, d_p)$  then  $x_n \rightarrow x$  in  $(X, p)$ . Let  $c \gg 0$  be given. Take  $q \in S$  and  $\delta > 0$  such that  $q(b) < \delta$  implies  $b \ll c$ . Since  $\frac{e}{n} \rightarrow 0$  in  $(E, S)$ , we can find  $\epsilon = \frac{1}{n_0}$  such that  $\epsilon q(e) = q(\epsilon e) < \delta$  and hence  $\epsilon e \ll c$ . Now, choose  $n_0$  such that  $d_p(x_n, x) = \xi_e \circ p(x_n, x) < \epsilon$  for all  $n \geq n_0$ . Hence, by Lemma 1.1 (iv) in [1],  $p(x_n, x) \ll \epsilon e \ll c$  for all  $n \geq n_0$ . The proof of the converse of implication (ii) is similar. Now it is possible to say that (iii) of Lemma 1 is immediate from the modified (i) and (ii). Thus, [[1], Theorem 2.2] should be complete as following.

**Theorem 2.** *Let  $(X, p)$  be a TVS-CMS,  $x \in X$  and  $\{x_n\}_{n=1}^{\infty}$  a sequence in  $X$ . Set  $d_p = \xi_e \circ p$ . Then the following statements hold:*

- (i)  $\{x_n\}_{n=1}^{\infty}$  converges to  $x$  in TVS-CMS  $(X, p)$  if and only if  $d_p(x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$ ,
- (ii)  $\{x_n\}_{n=1}^{\infty}$  is a Cauchy sequence in TVS-CMS  $(X, p)$  if and only if  $\{x_n\}_{n=1}^{\infty}$  is a Cauchy sequence in  $(X, d_p)$ ,
- (iii)  $(X, p)$  is a complete TVS-CMS if and only if  $(X, d_p)$  is a complete metric space.

**Remark 3.** *Above result says that for every complete TVS-cone metric space there exists a correspondent complete usual metric space such that the spaces are topologically isomorphic. Actually, by using the fact that for each  $c \gg 0$  there exist  $q \in S$  and  $\delta > 0$  such that  $q(b) < \delta$  implies that  $b \ll c$ , we can show that every TVS-cone metric space is a first countable topological space. This is of course possible if we assume that the cone has nonempty interior. However, there are still many interesting fixed point theorems that could be generalized to TVS-cone metric spaces such that their proofs do not follow directly by applying the nonlinear scalarization function  $\xi_e$  (see for example [3], [5], [6] and [7]).*

**Acknowledgment** The authors express their gratitude to the referee for constructive and useful remarks and suggestions.

## References

- [1] Du Wei-Shih: A note on cone metric fixed point theory and its equivalence *Nonlinear Analysis*, **72**(5), 2259-2261, (2010).
- [2] L. G. Huang and X. Zhang: Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, 332 (2007) 1468-1476.

- [3] Z. Kadelburg, S and Radenović and V. Rakoćević: Remarks on "quasi-contraction on a cone metric space", Appl. Math. Lett., 22 (2009), 1674-1679.
- [4] Sh. Rezapour and R. Hambarani: Some notes on the paper Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl., 345 (2008) 719-724.
- [5] Sh. Rezapour and R. H. Haghi and N. Shahzad: Some Notes on fixed points of quasicontraction maps, Appl. Math. Lett., 23 (2010), 498-502.
- [6] Sh. Rezapour, H. Khandani and S. M. Vaezpour: Efficacy of Cones on Topological Vector Spaces and Application to Common Fixed Points of Multifunctions, Rend. Circ. Mat. Palermo, 59 (2010), 185197.
- [7] Karapinar, E.: Fixed Point Theorems in Cone Banach Spaces, *Fixed Point Theory Appl.* 2009, Article ID 609281, 9 pages doi:10.1155/2009/609281.